

ON SPACES OF TOPOLOGICAL COMPLEXITY TWO

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ABSTRACT. In this paper we consider the classification of minimal cellular structures of spaces of topological complexity two under some hypotheses on there graded cohomological algebra. This continues the method used by M.Grant et al. in [1].

1. INTRODUCTION

The *topological complexity* (TC) of a space, introduced by M. Farber, is a numerical homotopy invariant analogous to the Lusternik-Schnirelmann category (cat) of that space. The two are expressed in terms of sectional category of a specific fibration (see §2). For instance, $cat(\mathbb{S}^n)$ equals to one either n is odd or even, but $TC(\mathbb{S}^n)$ equals one if n is odd and two if it is even. The goal of this paper is to investigate spaces with TC equals two, continuing the process begun by M. Grant et al. in [1].

Denote by X a non acyclic path-connected and finite type CW-complex (see below). According to Propositions 2.1 and Theorem 3.4.(A) in [1], and by the theorems of Hurewicz Whitehead :

If X is simply-connected and $\dim \tilde{H}^*(X, \mathbb{K}) = 1$ for all choices of field \mathbb{K} , then $TC(X) = 2$ if and only if X is of homotopy type of a sphere with even dimension.

In the sequel we will assume in addition that X is a non integral homology sphere. If X is simply-connected and not integral homology sphere. Denote $H_r(X)$, a non zero homology group of X , which occurs for an integer $r \geq 1$. Hypothesis implies that $H_r(X) \cong \mathbb{Z}^n \oplus T(H_r(X))$ ($T(H_r(X))$, being the torsion subgroup of $H_r(X)$), with the condition: if $n = 1$, we must have $T(H_r(X)) \neq 0$ or else, there exist another integer $s > r$ such that $H_s(X) \neq 0$. By the universal coefficient theorem, $H^r(X, \mathbb{Z}) \cong \text{Hom}(H_r(X), \mathbb{Z}) \neq 0$. We still use the identification $H_r(X) = \mathbb{Z}^n \oplus T(H_r(X))$.

We call an element u a *t-specific generator* if it is the unique generator contained in the free part of $H^t(X, \mathbb{Z})$, the non zero cohomology group with coefficients in \mathbb{Z} .

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Our first result states that the condition $TC(X) = 2$ influences algebra's structure of $H^*(X, \mathbb{K})$, for any field \mathbb{K} . Indeed, suppose $a \in H^r(X, \mathbb{Z})$ is a r -specific generator. Two cases are in the list:

- i) Suppose that $a^2 \neq 0$, then, for any field \mathbb{K} , $\dim \tilde{H}^*(X, \mathbb{K}) = 2$.
- ii) Suppose that there is an integer $s > r$ and a s -specific generator $b \in H^s(X, \mathbb{Z})$ such that the cup product $ab \neq 0$, then, for any field \mathbb{K} , $\dim \tilde{H}^*(X, \mathbb{K}) = 3$.

In the second step of our investigation of spaces having Topological Complexity equal two, we focus on the determination of the cells contained in CW-structure of X for each one of the two cases. To this end, we explicit the homology structure $H_*(X, \mathbb{Z})$, assuming (for any field \mathbb{K}) first, $\dim \tilde{H}^*(X, \mathbb{K}) = 2$ (cf. Theorem 1) and second, $\dim \tilde{H}^*(X, \mathbb{K}) = 3$ (cf. Theorem 2).

Theorem 1. (corollaries 7, 8). *Suppose that $TC(X) = 2$ and there exist a r -specific generator $a \in H^r(X, \mathbb{Z})$ such that $a^2 \neq 0$ then $\dim \tilde{H}^*(X, \mathbb{K}) = 2$ for all choices of field \mathbb{K} . In addition, X is of homotopy type of a space which contains only cells of dimensions r and $2r$.*

In the second case

Theorem 2. (corollaries 9, 11). *Suppose that $TC(X) = 2$ and there exist a r -specific generator $a \in H^r(X, \mathbb{Z})$, and a s -specific generator $b \in H^s(X, \mathbb{Z})$ such that $ab \neq 0$; then $\dim \tilde{H}^*(X, \mathbb{K}) = 3$ for all choices of field \mathbb{K} . In addition, X is of homotopy type of a CW-complex which contain only cells of dimensions r , s and $(r + s)$.*

2. DEFINITION AND PRELIMINARY RESULTS

The sectional category of a fibration $p : E \rightarrow B$, denoted by $secat(p)$, is the smallest number n for which there is an open covering $\{U_0, \dots, U_n\}$ of B by $(n + 1)$ open sets, for each of which there is a local section $s_i : U_i \rightarrow E$ of p , so that $p \circ s_i = j_i : U_i \rightarrow B$, where j_i denotes the inclusion.

Let X^I , where $I = [0, 1]$ denote the space of (free) paths on a space X . there is a fibration, substitute from the diagonal map $\Delta : X \rightarrow X \times X$, $\pi : X^I \rightarrow X \times X$, which evaluates a continuous path at initial and final point, i.e. for $\alpha \in X^I$, we have $\pi(\alpha) = (\alpha(0), \alpha(1))$.

We define the topological complexity (normalized) of X , noted $TC(X)$, to be the sectional category $secat(\pi)$ of this fibration. That is, $TC(X)$ is the smallest number n for which there is an open cover $\{U_0, \dots, U_n\}$ of $X \times X$ by $(n + 1)$ open sets, for each of which there is a local section $s_i : U_i \rightarrow X^I$ of π , i.e., for which $\pi \circ s_i = j_i : U_i \rightarrow X \times X$, where j_i denotes the inclusion. For instance, $TC(S^n)$ equal 1 for n odd and 2 for n even, e.g. $TC(T^1) = 2$ where $T^2 = S^1 \times S^1$, is the tore.

Recall also that X is of finite type if $H_r(X)$ is a finite integral homology group for any integer r .

Definition 3. *Let \mathbb{K} be a field. the homomorphism induced on cohomology with coefficients in \mathbb{K} by the diagonal map $\Delta : X \rightarrow X \times X$ (and thus by $P_2 : PX \rightarrow X \times X$ which is a*

fibration substitute for it) may be identified with the cup product homomorphism

$$U_2(X) : H^*(X, \mathbb{K}) \otimes H^*(X, \mathbb{K}) \longrightarrow H^*(X, \mathbb{K})$$

The ideal of zero divisors is $\ker U_2(X)$, the kernel of $U_2(X)$, the zero-divisors cup length is $\text{nil}(\ker U_2(X))$, the nilpotency of this ideal. which is to say the number of factors in the longest non-trivial product of elements from this ideal.

We denote $U_2(X)(a \otimes b)$ and $a \smile b$ by ab (taking into consideration the system of coefficients).

Proposition 4. ([4, Thm 7]) For any field \mathbb{K} , we have

$$\text{nil}(\ker U_2(X)) \leq TC(X).$$

The cup product can be defined for more general coefficient groups. For example, the Alexander-Whitney diagonal approximation gives a cup product

$$\smile : H^p(X; G_1) \otimes H^q(X; G_2) \longrightarrow H^{p+q}(X; G_1 \otimes G_2),$$

by putting $(a \smile b)(\sigma) = a(\sigma/[v_0, \dots, v_p]) \otimes b(\sigma/[v_p, \dots, v_{p+q}])$. In particular, since $\mathbb{Z} \otimes \mathbb{Z}/k\mathbb{Z} \cong \mathbb{Z}/k\mathbb{Z}$, there is the cup product $H^p(X; \mathbb{Z}) \otimes H^q(X; \mathbb{Z}/k\mathbb{Z}) \longrightarrow H^{p+q}(X; \mathbb{Z}/k\mathbb{Z})$ and it coincides with the product obtained by first reducing mod k and then taking the cup product over the ring $\mathbb{Z}/k\mathbb{Z}$. This also the case for any field K . We will denote it by:

$$\smile_1 : H^p(X; \mathbb{Z}) \otimes H^q(X; K) \longrightarrow H^{p+q}(X; K)$$

3. PROOFS OF OUR RESULTS

Let X be a path connected CW-complex not acyclic and nor homology sphere. Let r , denotes, the smallest integer such that $H^r(X, \mathbb{Z})$ is not trivial.

Theorem 5. Let n be an integer such that $n \geq 2$ so :

- (1) If $TC(X) = n$ and there exist a r -specific generator $a \in H^r(X, \mathbb{Z})$ such that $a^n \neq 0$, then, $\dim \tilde{H}^*(X, \mathbb{K}) = n$ for all choices of field \mathbb{K} . Furthermore, X is of homotopy type of a CW-complex which contains only cells of dimension $r, 2r, \dots$ and nr .
- (2) If $TC(X) = n$ and suppose that there exist $\{a_i/1 \leq i \leq n\}$ a finite set of r_i -specific generators (with $r_1 < r_2 < \dots < r_n$) such that $a_1 a_2 \dots a_n \neq 0$, then $\dim \tilde{H}^*(X, \mathbb{K}) = n+1$ for all choices of field \mathbb{K} . Furthermore, X is of homotopy type of a CW-complex which contains only cells of dimensions r_1, r_2, \dots, r_n and $r_1 + r_2 + \dots + r_n$.

Proof. Let $n \geq 2$.

- (1) Let \mathbb{K} be a field and $1_{\mathbb{K}} \in H^0(X, \mathbb{K})$ be the unit element, we have from the hypothesis a, a^2, \dots, a^n are nonzero so $a \smile_1 1_{\mathbb{K}}, a^2 \smile_1 1_{\mathbb{K}}, \dots, a^n \smile_1 1_{\mathbb{K}}$ are nonzero, hence $\dim \tilde{H}^*(X, \mathbb{K}) \geq n$.

Suppose now that there exist $b \in H^s(X, \mathbb{K})$ such that $a \smile_1 1_{\mathbb{K}}, a^2 \smile_1 1_{\mathbb{K}}, \dots, a^n \smile_1 1_{\mathbb{K}}$ and b are linearly independent over \mathbb{K} so we can proof by induction that $((a \smile_1 1_{\mathbb{K}}) \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes (a \smile_1 1_{\mathbb{K}}))^n (b \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes b)$ contains $(a^n \smile_1 1_{\mathbb{K}}) \otimes b \pm b \otimes (a^n \smile_1 1_{\mathbb{K}})$

which is nonzero because $a^n \smile_1 1_{\mathbb{K}}$ and b are linearly independent; and since $|a| \neq |b|$ hence $((a \smile_1 1_{\mathbb{K}}) \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes (a \smile_1 1_{\mathbb{K}}))^n (b \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes b)$ is nonzero, then from *proposition 4* we have $TC(X) \geq \text{nilker}(U_2(X)) \geq n+1$ which is a contradiction. Therefore $\dim \tilde{H}^*(X, \mathbb{K}) = n$.

Suppose that $\exists s \notin \{r, 2r, \dots, nr\}$ such that $H_s(X) \neq 0$ (where $s \geq 1$).

if $\text{Free}(H_s(X)) \supseteq \mathbb{Z}$ we have for $\mathbb{K} = \mathbb{Q}$ and from the Universal Coefficient Theorem (*UCT*) $H^s(X, \mathbb{Q}) \supseteq \mathbb{Q}$ since $H^r(X, \mathbb{Q}) \neq 0$, $H^{2r}(X, \mathbb{Q}) \neq 0$, \dots , $H^{nr}(X, \mathbb{Q}) \neq 0$ hence $\dim \tilde{H}^*(X, \mathbb{Q}) \geq n+1$ which is a contradiction. else if $\mathbb{Z}/p^k\mathbb{Z} \subset T(H_s(X)) \neq 0$, where p is a prime number, for $K = \mathbb{Z}/p\mathbb{Z}$ we have from the *UCT* $H^{s+1}(X, \mathbb{Z}/p\mathbb{Z}) \supseteq \text{Ext}(H_s(X), \mathbb{Z}/p\mathbb{Z}) \supseteq S \neq 0$ (where S is a $\mathbb{Z}/p\mathbb{Z}$ -vector space) since $H^r(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$, $H^{2r}(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$, \dots , $H^{nr}(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$ hence $\dim \tilde{H}^*(X, \mathbb{Z}/p\mathbb{Z}) \geq n+1$ which is a contradiction.

Therefor $H_s(X) = 0$ for all $s \notin \{r, 2r, \dots, nr\}$.

Suppose that $\text{free}(H_r(X)) = 0$ so from the *UCT* we have $0 \neq H^r(X, \mathbb{Z}) \cong \text{Free}(H_r(X)) \oplus T(H_{r-1}(X))$ then $\tilde{H}_{r-1}(X) \neq 0$ which is a contradiction from the previous result.

Suppose now that $H_t(X) = \mathbb{Z}^m \oplus T(H_t(X))$ for $t \in \{r, 2r, \dots, nr\}$ (where $t \geq 1$). From the *UCT* we have $H^t(X, \mathbb{Q}) \supseteq \mathbb{Q}^m$, if $m \geq 2$ then $\dim \tilde{H}^*(X, \mathbb{Q}) \geq n+1$ which is a contradiction. Hence $H_t(X) = \mathbb{Z}^m \oplus T(H_t(X))$ where $m \in \{0, 1\}$. Also if $\mathbb{Z}/p^k\mathbb{Z} \subset T(H_t(X))$ so by the *UCT* for $\mathbb{K} = \mathbb{Z}/p\mathbb{Z}$, we have $H^{t+1}(X, \mathbb{Z}/p\mathbb{Z}) \supseteq \mathbb{Z}/p\mathbb{Z} \oplus T$ (where T is a $\mathbb{Z}/p\mathbb{Z}$ -vector space) then $\dim \tilde{H}^*(X, \mathbb{K}) \geq n+1$ which is a contradiction. Finally $H_t(X) = \mathbb{Z}$ for all $t \in \{r, 2r, \dots, nr\}$.

Then X is of homotopy type of a CW-complex which contains only cells of dimensions $r, 2r, \dots$, and nr .

- (2) Let \mathbb{K} be a field and $1_{\mathbb{K}} \in H^0(X, K)$ be the unit element, Clearly, $\dim \tilde{H}^*(X, \mathbb{K}) \geq n+1$. Suppose now that there exist $b \in H^s(X, \mathbb{K})$ such that $a_1 \smile_1 1_{\mathbb{K}}, a_2 \smile_1 1_{\mathbb{K}}, \dots, a_n \smile_1 1_{\mathbb{K}}, (a_1 \smile_1 1_{\mathbb{K}})(a_2 \smile_1 1_{\mathbb{K}}) \dots (a_n \smile_1 1_{\mathbb{K}})$ and b are linearly independent over K . We can proof by induction that the term $((a_1 \smile_1 1_{\mathbb{K}}) \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes (a_1 \smile_1 1_{\mathbb{K}}))((a_2 \smile_1 1_{\mathbb{K}}) \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes (a_2 \smile_1 1_{\mathbb{K}})) \dots ((a_n \smile_1 1_{\mathbb{K}}) \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes (a_n \smile_1 1_{\mathbb{K}}))(b \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes b)$ contains $(a_1 \smile_1 1_{\mathbb{K}})(a_2 \smile_1 1_{\mathbb{K}}) \dots (a_n \smile_1 1_{\mathbb{K}}) \otimes b \pm b \otimes (a_1 \smile_1 1_{\mathbb{K}})(a_2 \smile_1 1_{\mathbb{K}}) \dots (a_n \smile_1 1_{\mathbb{K}})$ which is nonzero because $(a_1 \smile_1 1_{\mathbb{K}})(a_2 \smile_1 1_{\mathbb{K}}) \dots (a_n \smile_1 1_{\mathbb{K}})$ and b are linearly independents over K ; since $|b| \neq |a_i| \forall i \in I$ so $((a_1 \smile_1 1_{\mathbb{K}}) \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes (a_1 \smile_1 1_{\mathbb{K}}))((a_2 \smile_1 1_{\mathbb{K}}) \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes (a_2 \smile_1 1_{\mathbb{K}})) \dots ((a_n \smile_1 1_{\mathbb{K}}) \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes (a_n \smile_1 1_{\mathbb{K}}))(b \otimes 1_{\mathbb{K}} - 1_{\mathbb{K}} \otimes b)$ is nonzero. Then from *proposition 4* we have $TC(X) \geq \text{nilker}(U_2(X)) \geq n+2$ which is a contradiction. Therefor $\dim \tilde{H}^*(X, \mathbb{K}) = n+1$.

Suppose now that $\exists s \notin \{r_1, r_2, \dots, r_n, r_1 + \dots + r_n\}$ (where $s \geq 1$) such that $H_s(X) \neq 0$.

If at least $H_s(X) \supseteq \mathbb{Z}$ so we have from the *UCT* $H^s(X, \mathbb{Q}) \supseteq \mathbb{Q}$, since $H^{r_i}(X, \mathbb{Q}) \neq 0$

for $i \in I$ and $H^{r_1+\dots+r_n}(X, \mathbb{Q}) \neq 0$ hence $\dim \tilde{H}^*(X, \mathbb{Q}) \geq n+2$ which is a contradiction.

Also if $T(H_s(X)) \neq 0$ so there exist at least a non zero summand $\mathbb{Z}/p^k\mathbb{Z}$, where p is a prime number, so from the *UCT* we have $H^s(X, \mathbb{Z}/p\mathbb{Z}) \supseteq \mathbb{Z}/p\mathbb{Z} \oplus S$ and $H^{s+1}(X, \mathbb{Z}/p\mathbb{Z}) \supseteq \mathbb{Z}/p\mathbb{Z} \oplus T$ (where S and T are two $\mathbb{Z}/p\mathbb{Z}$ -vector spaces), since $H^{r_i}(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$ for $i \in I$ and $H^{r_1+\dots+r_n}(X, \mathbb{Z}/p\mathbb{Z}) \neq 0$ then $\dim \tilde{H}^*(X, \mathbb{Z}/\mathbb{Z}) \geq n+3$ which is also a contradiction.

So for all integer $s \notin \{r_1, \dots, r_n, r_1 + \dots + r_n\}$, $H_s(X)$ is trivial.

As in the proof of the first part in the theorem and from the result above we have $\text{Free}(H_r(X)) \neq 0$.

Suppose that $\exists t \in \{r_1, \dots, r_n, r_1 + \dots + r_n\}$ such that $H_t(X) \cong \mathbb{Z}^n \oplus T(H_t(X))$ (where $t \geq 1$). If $T(H_t(X))$ has at least a summand $\mathbb{Z}/p^k\mathbb{Z}$ and $n \geq 1$ so by the *UCT* we have $H^t(X, \mathbb{Z}/p\mathbb{Z}) \supseteq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus S$, where S is a $\mathbb{Z}/p\mathbb{Z}$ -vector space, hence $\dim \tilde{H}^*(X, \mathbb{Z}/p\mathbb{Z}) \geq n+2$ which is a contradiction then $n = 0$ or $T(H_t(X)) = 0$. Suppose now that $T(H_t(X))$ has at least a summand $\mathbb{Z}/p^k\mathbb{Z}$ hence $n = 0$, since $H^t(X, \mathbb{Z}) = \text{Free}(H_t(X)) \oplus T(H_{t-1}(X))$ so $T(H_{t-1}(X)) \neq 0$ if $t-1 \notin \{r_1, \dots, r_n, r_1 + \dots + r_n\}$ it's a contradiction else (i.e. $t-1 \in \{r_1, \dots, r_n, r_1 + \dots + r_n\}$) we proceed in the same manner and we will have $T(H_{t-2}(X)) \neq 0$ we continue the process till to get an element not in $\{r_1, \dots, r_n, r_1 + \dots + r_n\}$ which is a contradiction.

Therefor $n = 1$ and $T(H_t(X)) = 0$ i.e. $H_t(X) \cong \mathbb{Z}$, $\forall t \in \{r_1, \dots, r_n, r_1 + \dots + r_n\}$. Then X is of homotopy type of a CW-complex which contains only cells of dimensions r_1, r_2, \dots, r_n , and $r_1 + r_2 + \dots + r_n$

□

Remark 6. (1) *The results of the previous theorem holds true if $|a_1| = \dots = |a_n| = r$ and a_1, \dots, a_n are the unique generators of degree r .*

(2) *Since the cup product of the wedge of spheres is trivial, the space X which verify the hypothesis of the theorem 5 won't necessary be wedge of spheres. In our next work, we will focus on determining homotopy types of attaching maps in minimal structures for various cases cited in the previous theorem*

Applying *theorem 5* in the case of topological complexity two, we have immediately the :

Corollary 7. *Suppose that $TC(X) = 2$ and there exist a r -specific generator $a \in H^r(X, \mathbb{Z})$ such that $a^2 \neq 0$ hence $\dim \tilde{H}^*(X, K) = 2$ for all field K .*

Corollary 8. *With the same hypothesis as in corollary 7, we have $H_r(X) = \mathbb{Z}$, $H_{2r}(X) = \mathbb{Z}$ and $H_s(X) = 0$ for all $s \notin \{r, 2r\}$, Furthermore X is of homotopy type of a CW-complex which contains only cells of dimensions r and $2r$*

Corollary 9. *Suppose that $TC(X) = 2$ and there exist a r -specific generator $a \in H^r(X, \mathbb{Z})$, and a s -specific generator $b \in H^s(X, \mathbb{Z})$ (where $r < s$) such that $ab \neq 0$; therefore $\dim \tilde{H}^*(X, K) = 3$ for all field K .*

The previous corollary holds true if $|a| = |b| = r$ and are the unique generators of degree r .

As an example we take the case of the Tore :

Example 10. *If $X = T^2$. we have $TC(T^2) = 2$ and there exist two generators a and b which they generate $H^1(X, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ such that ab generates $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$, so $\dim \tilde{H}^*(X, K) = 3$ for all field K .*

Corollary 11. *with the same properties as in corollary 9, we have X is of homotopy type of a CW-complex which contains only cells of dimensions r , s and $r + s$*

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